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Parametrization of SU(3) orbits in E_8

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Abstract. Using Michel and Radicati's description of the stratification of E_8 under the action of SU(3) we construct a parametrization of the SU(3) adjoint representation in terms of invariants and angles. This is the SU(3) equivalent of the SU(2) change from cartesian coordinates to spherical polars.

1. Introduction

In theories involving nonlinear realizations of chiral symmetries on meson fields we are involved with constructing nonlinear, and often nonpolynomial, functions of the fields. These functions are defined through their power series expansions but the field theory is generally not well defined. The mesons are pseudoscalar field operators (more properly operator valued distributions) and consequently we have all the infinity problems of products of field operators at the same space–time point.

Nevertheless, in order to use normal functional methods in constructing such objects as covariant derivatives (Gasiorowicz and Geffen 1969 and references therein) and superpropagators (Delbourgo 1970) (if the Lagrangian is nonpolynomial) we treat the field multiplets as real, c number quantities; that is, as coordinates in a real euclidian manifold. This allows us to differentiate, integrate and form power series expansions with respect to the fields.

We are concerned in this paper with the group SU(3). Because the action of the group on the pseudoscalar mesons M_i (as usual these are assigned to the adjoint representation) defines Casimir invariants, and because we are often dealing with functions which are constructed only of the invariants, a change of coordinates from the M_i to a set of invariants and angles is desirable. This is particularly useful in handling integral transforms of invariant functions of the field multiplets, as exhibited for SU(2) in a recent paper by Delbourgo (1970).

For real orthogonal groups and SU(2) with its triplet adjoint representation the change of coordinates is well known, being from cartesian to spherical polars. For SU(3), however, where the adjoint group SU(3)/Z(3) is isomorphic only to a subgroup of a real orthogonal group, the equivalent change of coordinates with its associated Jacobian is more complicated.

In § 2 we review the simple SU(2) case in a rather complex language but this serves to illustrate the techniques we use in §§ 3 and 4 for SU(3).

2. The SU(2) stratification and parametrization of E_3

The action of SU(2) on the real three dimensional euclidean manifold E_3, cartesian coordinates $\{\pi_i\}$, is realized via the triplet matrices $\boldsymbol{\pi} = \pi_i \boldsymbol{\sigma}_i$. The $\boldsymbol{\sigma}_i$ are the Pauli 2×2 traceless hermitian matrices with the product law

$$\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j = \delta_{ij} + \mathrm{i}\epsilon_{ijk}\boldsymbol{\sigma}_k .$$

The matrices $\boldsymbol{\pi}$ belong to the selfrepresentation of SU(2), and

$$\boldsymbol{\pi}' = \pi_i' \boldsymbol{\sigma}_i = \boldsymbol{U}\boldsymbol{\pi}\boldsymbol{U}^{-1} = \pi_i \boldsymbol{U}\boldsymbol{\sigma}_i \boldsymbol{U}^{-1}$$

for $\boldsymbol{U} \in \mathrm{SU}(2)$. From this we can establish the well known isomorphism of the adjoint group SU(2)/Z(2) with SO(3), so the action on vectors is length preserving. Equivalently we could consider the SU(2) transformations leaving invariant the characteristic equation

$$\boldsymbol{\pi}^2 - \beta(\pi)\mathbf{1} = 0$$

where $\beta(\pi) = \frac{1}{2}\mathrm{Tr}(\boldsymbol{\pi}\boldsymbol{\pi}) = (\pi_i)^2$. The action of the group partitions the manifold E_3 into orbits† of constant β. As $\boldsymbol{\pi}$ is hermitian it has real eigenvalues, hence $\beta \geqslant 0$ and the two cases, $\beta > 0$ and $\beta = 0$, distinguish for us two types of orbit. There is the null orbit with the little group the whole of SU(2), and the orbits of positive-definite β (spheres); the little group in SU(2) being U(1). These are respectively of dimension‡ 0 and 2. If we define a stratum (Michel 1968) as being a set of points having the same little group up to a conjugation, then the action of SU(2) on E_3 can be said to partition it into two strata, each stratum being partitioned into orbits of the same type. A *general* point in the space must then include representatives from each stratum as this decomposition is group invariant, that is, we cannot go from one stratum to another by the action of the group. This is rather simple in this case as one stratum is the origin. We are just saying that a general $\boldsymbol{\pi}$ can be represented by

$$\boldsymbol{\pi} = a\boldsymbol{\eta} + b\boldsymbol{O}$$

where $\boldsymbol{\eta}$ has positive definite normalization. If we take this to be unity, $\frac{1}{2}\mathrm{Tr}(\boldsymbol{\eta}\boldsymbol{\eta}) = 1$, then a is readily identified as $\sqrt{\beta}\,(\equiv r)$. If we pick an $\boldsymbol{\eta}^0$ in a particular direction then

$$\pi_i \boldsymbol{\sigma}_i = r\boldsymbol{U}\eta_i^0 \boldsymbol{\sigma}_i \boldsymbol{U}^{-1}$$

where $\boldsymbol{U}$ is a general element of SU(2). Thus we have a parametrization of the π_i through

$$\pi_i = r\tfrac{1}{2}\mathrm{Tr}(\boldsymbol{\sigma}_i \boldsymbol{U}\boldsymbol{\sigma}_j \boldsymbol{U}^{-1})\eta_j^0 .$$

In particular if we choose $\boldsymbol{\eta}^0$ to be diagonal, $\eta_i^0 = \delta_{i3}$, and take the eulerian form for a general $\boldsymbol{U}$

$$\boldsymbol{U} = R_3(\phi)R_2(\theta)R_3(\psi)$$

where $R_k(\alpha) = \exp(-\frac{1}{2}\mathrm{i}\alpha\boldsymbol{\sigma}_k)$, we recover the usual spherical polar form

$$\pi_1 + \mathrm{i}\pi_2 = r\sin\theta\, \mathrm{e}^{\mathrm{i}\phi}$$

$$\pi_3 = r\cos\theta .$$

† The orbit of π is the set of all $U\pi U^{-1}$ transformed from π by all the U of the group. The orbits are either identical or disjoint, hence the partitioning by the group.

‡ If G is a Lie group acting differentiably on a manifold M and if H is the little group (closed in G) of an orbit, then that orbit is a submanifold of M whose dimension is: (dim G − dim H).

We note that for $r > 0$ the orbits are two dimensional as expected, our choice of $\boldsymbol{\eta}^0$ having removed the ψ dependence. Trivially, for $r = 0$, the orbit is just the origin. The Jacobian for the change of coordinates is $r^2 \sin\theta$.

3. The SU(3) stratification of E_8

We now proceed to construct the equivalent parametrization for the eight components of the SU(3) adjoint representation M_i. The action of the group on the manifold E_8 is realized via the octet matrices $\boldsymbol{M} = M_i \boldsymbol{\lambda}_i$, where the $\boldsymbol{\lambda}_i$ are the Gell-Mann 3×3 traceless, hermitian matrices with the product law

$$\boldsymbol{\lambda}_i \boldsymbol{\lambda}_j = \tfrac{2}{3}\delta_{ij} + (d_{ijk} + \mathrm{i} f_{ijk})\boldsymbol{\lambda}_k .$$

Michel and Radicati (1968) have studied the stratification of this matrix space under SU(3) and have shown that we have a partitioning into three strata. We give a very brief review of their results.

The characteristic equation of the $\boldsymbol{M}$ is

$$\boldsymbol{M}^3 - \boldsymbol{M}\gamma(M) - \mu(M) = 0$$

where

$$\begin{aligned} \gamma(M) &= \tfrac{1}{2}\,\mathrm{Tr}(\boldsymbol{MM}) = (M_i)^2 \\ \mu(M) &= \tfrac{1}{3}\,\mathrm{Tr}(\boldsymbol{MMM}) = \det \boldsymbol{M} = \tfrac{2}{3} d_{ijk} M_i M_j M_k . \end{aligned} \tag{1}$$

As $\boldsymbol{M}$ is hermitian we expect three real eigenvalues and the condition for this is

$$4\gamma^3 \geqslant 27\mu^2 \geqslant 0. \tag{2}$$

Any $\boldsymbol{M}$ can be diagonalized by an element of SU(3), hence the orbits are exactly labelled by the two invariants γ and μ. The above inequalities distinguish the three strata.

(i) *Generic stratum*

If $4\gamma^3 > 27\mu^2$, $\boldsymbol{M}$ has three distinct eigenvalues, their sum must be zero, and the little group in SU(3) of such a diagonal matrix is $\mathrm{U}(1) \times \mathrm{U}(1)$. This is of dimension 2 so that the orbits are of dimension $8-2 = 6$. This stratum is thus a two parameter family of six dimensional orbits.

(ii) *Special stratum*

If $4\gamma^3 = 27\,\mu^2 > 0$, we have a repeated eigenvalue, and the little group in SU(3) is U(2). The dimension of the orbits is thus $8-4 = 4$ and so we have a one parameter family of four dimensional orbits.

Because of the repeated eigenvalue, such $\boldsymbol{M}$ satisfy a second degree equation which can be identified as

$$(d_{ijk} M_i M_j)\boldsymbol{\lambda}_k = (\tfrac{1}{2}\mu(M))^{1/3}\boldsymbol{M}. \tag{3}$$

This is the only property of the special stratum that we shall use, but we would remark to the interested reader that elements of this stratum are very important to physics. Michel and Radicati suggestively name these *charge* elements and study their properties in detail.

(iii) *Null stratum*

If $4\gamma^3 = 27\mu^2 = 0$, we have the trivial zero dimensional origin with little group the whole of SU(3).

If we are now to construct a general point of the matrix space we know it must have the form

$$\boldsymbol{M} = a\boldsymbol{Z}+b\boldsymbol{Q}+c\boldsymbol{O}$$

where $\boldsymbol{Z}\in$ generic stratum, $\boldsymbol{Q}\in$ charge stratum. The coefficients a, b (and c for completeness) are functions of the invariants γ and μ. If we choose a $\boldsymbol{U}\in \mathrm{SU}(3)$ such that $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{M}^0\boldsymbol{U}^{-1}$ for $\boldsymbol{M}^0$ diagonal

$$\boldsymbol{M} = \boldsymbol{U}(a\boldsymbol{Z}^0+b\boldsymbol{Q}^0)\boldsymbol{U}^{-1}.$$

Now let us consider the diagonal charge $\boldsymbol{Q}^0$. It must be of the form $(g\boldsymbol{\lambda}_3+h\boldsymbol{\lambda}_8)$ and, applying the charge condition (3), gives $g = 0$, or $g = \pm\sqrt{3}\,h$ for any h. We will take the choice of $g = 0$, and will normalize with $h = 1$. The reason is that, with the particular form of the $\boldsymbol{U}$ we shall choose, the four dimensional nature of the charge orbits will be simply seen. Having thus chosen $\boldsymbol{Q}^0$ as $\boldsymbol{\lambda}_8$, we take a diagonal generic element as $\boldsymbol{\lambda}_3$. Then the generic and charge elements are orthogonal, $\frac{1}{2}\,\mathrm{Tr}(\boldsymbol{Z}\boldsymbol{Q}) = 0$.

Finally, if the projections of $\boldsymbol{M}$ onto the generic and charge strata are given by $a = m\sin\omega$ and $b = -m\cos\omega$, we can use equations (1) to identify

$$m = +\sqrt{\gamma}\,(M) \qquad \text{and} \qquad \cos 3\omega = \frac{3\sqrt{3}}{2}\mu(M)m^{-3}.$$

The condition (2) ensures $|\cos 3\omega| \leqslant 1$, hence ω is a real angle.

We thus have, in the octet matrix space

$$\boldsymbol{M} = m\sin\omega\boldsymbol{Z}-m\cos\omega\boldsymbol{Q}$$

and so the general E_8 vector is given by

$$M_i = m\sin\omega Z_i - m\cos\omega Q_i \tag{4}$$

where the generic vector

$$Z_i = \tfrac{1}{2}\,\mathrm{Tr}(\boldsymbol{\lambda}_i\boldsymbol{U}\boldsymbol{\lambda}_3\boldsymbol{U}^{-1}) \tag{5a}$$

and the orthogonal charge vector

$$Q_i = \tfrac{1}{2}\,\mathrm{Tr}(\boldsymbol{\lambda}_i\boldsymbol{U}\boldsymbol{\lambda}_8\boldsymbol{U}^{-1}). \tag{5b}$$

In the next section we explicitly construct these vectors in terms of the angular parameters of $\boldsymbol{U}$.

In obtaining the above form of M_i we have only used the fact that a general $\boldsymbol{M}$ must have projections into each stratum. With pedagogic intent we remark that in fact our particular decomposition of $\boldsymbol{M}$ is an example of a stronger result due to Michel and Radicati (1969 'The geometry of the octet', unpublished). They show that, if $\boldsymbol{S}$ is a generic element of the octet matrix space with $\gamma(s) = 1$ and $\mu(s) = 0$, then the element

$$\boldsymbol{Q}(s) \equiv (d_{ijk}S_iS_j)\boldsymbol{\lambda}_k$$

is an associated charge element orthogonal to $\boldsymbol{S}$. Furthermore, an $\boldsymbol{S}$ can always be chosen such that for any $\boldsymbol{M}$ we have $\boldsymbol{M} = \alpha\,.\,\boldsymbol{S}+\beta\,.\,\boldsymbol{Q}(s)$.

Our generic vector, $\boldsymbol{Z} = \boldsymbol{U}\boldsymbol{\lambda}_3\boldsymbol{U}^{-1}$, is a particular example of such an $\boldsymbol{S}$, and $\boldsymbol{Q} = \boldsymbol{U}\boldsymbol{\lambda}_8\boldsymbol{U}^{-1} = \sqrt{3}\,\boldsymbol{Q}(z)$.

4. SU(3) angular parametrization of E_8

All that remains to do now is to put an explicit eight-parameter general $U \in \mathrm{SU}(3)$ into equations (5) and thus construct the vectors Z_i and Q_i. We can also construct, from equation (4), the Jacobian of the transformation from the cartesian M_i to the invariants and angles.

We choose a pseudoeulerian form for U, of the type

$$\begin{bmatrix}\text{general } I \text{ spin}\\ \text{rotation}\end{bmatrix}\begin{bmatrix}\text{hypercharge changing}\\ \text{rotation}\end{bmatrix}\begin{bmatrix}\text{general I spin}\\ \text{rotation}\end{bmatrix}\begin{bmatrix}\text{hypercharge}\\ \text{rotation}\end{bmatrix}$$

In terms of eight angles this can be taken as (Nelson 1967)

$$U = [T_3(\theta_1)T_2(\theta_2)T_3(\theta_3)]T_7(\theta_4)[T_3(\theta_5)T_2(\theta_6)T_3(\theta_7)]T_8(\theta_8)$$

where

$$T_k(\theta_l) = \exp(-\tfrac{1}{2}\mathrm{i}\theta_l\lambda_k)$$

and we immediately see that as λ_8 commutes with T_2, T_3 and T_8 then the charge vector Q_i given by equation (5*b*) is four dimensional, depending only on the first four angles. In fact, the four parameter $U' = T_3T_2T_3T_8$ forms a representation of the U(2) little group of $\boldsymbol{Q}^0$ in SU(3).

Similarly the generic vector Z_i is six dimensional with the $U'' = T_3T_8$ forming a representation of the $\mathrm{U}(1)\times\mathrm{U}(1)$ little group of $\boldsymbol{Z}^0$.

The results for the charge vector are

$$\begin{aligned}
Q_1+\mathrm{i}Q_2 &= \tfrac{1}{2}\sqrt{3}\sin\theta_2\sin^2\tfrac{1}{2}\theta_4\,\mathrm{e}^{\mathrm{i}\theta_1}\\
Q_3 &= \tfrac{1}{2}\sqrt{3}\cos\theta_2\sin^2\tfrac{1}{2}\theta_4\\
Q_4+\mathrm{i}Q_5 &= -\tfrac{1}{2}\sqrt{3}\sin\tfrac{1}{2}\theta_2\sin\theta_4\exp\{\tfrac{1}{2}\mathrm{i}(\theta_1-\theta_3)\}\\
Q_6+\mathrm{i}Q_7 &= \tfrac{1}{2}\sqrt{3}\cos\tfrac{1}{2}\theta_2\sin\theta_4\exp\{-\tfrac{1}{2}\mathrm{i}(\theta_1+\theta_3)\}\\
Q_8 &= \tfrac{1}{2}(3\cos^2\tfrac{1}{2}\theta_4-1)
\end{aligned}$$

and for the generic vector

$$\begin{aligned}
Z_1+\mathrm{i}Z_2 &= \tfrac{1}{2}\sin\theta_2\cos\theta_6(1+\cos^2\tfrac{1}{2}\theta_4)\,\mathrm{e}^{\mathrm{i}\theta_1}\\
&\quad+\cos^2\tfrac{1}{2}\theta_2\sin\theta_6\cos\tfrac{1}{2}\theta_4\exp\{\mathrm{i}(\theta_1+\theta_3+\theta_5)\}\\
&\quad-\sin^2\tfrac{1}{2}\theta_2\sin\theta_6\cos\tfrac{1}{2}\theta_4\exp\{\mathrm{i}(\theta_1-\theta_3-\theta_5)\}\\
Z_3 &= \tfrac{1}{2}\cos\theta_2\cos\theta_6(1+\cos^2\tfrac{1}{2}\theta_4)-\sin\theta_2\sin\theta_6\cos\tfrac{1}{2}\theta_4\cos(\theta_3+\theta_5)\\
Z_4+\mathrm{i}Z_5 &= \cos\tfrac{1}{2}\theta_2\sin\theta_6\sin\tfrac{1}{2}\theta_4\exp\{\tfrac{1}{2}\mathrm{i}(\theta_1+\theta_3+2\theta_5)\}\\
&\quad+\tfrac{1}{2}\sin\tfrac{1}{2}\theta_2\cos\theta_6\sin\theta_4\exp\{\tfrac{1}{2}\mathrm{i}(\theta_1-\theta_3)\}\\
Z_6+\mathrm{i}Z_7 &= \sin\tfrac{1}{2}\theta_2\sin\theta_6\sin\tfrac{1}{2}\theta_4\exp\{-\tfrac{1}{2}\mathrm{i}(\theta_1-\theta_3-2\theta_5)\}\\
&\quad-\tfrac{1}{2}\cos\tfrac{1}{2}\theta_2\cos\theta_6\sin\theta_4\exp\{-\tfrac{1}{2}\mathrm{i}(\theta_1+\theta_3)\}\\
Z_8 &= \tfrac{1}{2}\sqrt{3}\cos\theta_6\sin^2\tfrac{1}{2}\theta_4.
\end{aligned}$$

The Jacobian for the change of coordinates is $\frac{1}{32}m^7\sin^2 3\omega\sin\theta_6\sin\theta_2\sin\theta_4\sin^2\tfrac{1}{2}\theta_4$

and the angular ranges are

$$0 \leqslant \theta_1, \theta_3, \theta_5 < 2\pi$$

$$0 \leqslant \theta_2, \theta_4, \theta_6 \leqslant \pi.$$

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